Proper Actions of High-Dimensional Groups on Complex Manifolds*†

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We explicitly classify all pairs (M, G), where M is a connected complex manifold of dimension $n \geq 2$ and G is a connected Lie group acting properly and effectively on M by holomorphic transformations and having dimension d_G satisfying $n^2 + 2 \leq d_G < n^2 + 2n$. These results extend – in the complex case – the classical description of manifolds admitting proper actions of groups of sufficiently high dimensions. They also generalize some of the author's earlier work on Kobayashi-hyperbolic manifolds with high-dimensional holomorphic automorphism group.

0 Introduction

Let M be a connected C^∞-smooth manifold and $\mathrm{Diff}(M)$ the group of C^∞-smooth diffeomorphisms of M endowed with the compact-open topology. A topological group G is said to act continuously on M by diffeomorphisms, if a continuous homomorphism $\Phi : G \to \mathrm{Diff}(M)$ is specified. The continuity of Φ is equivalent to the continuity of the action map

$$\hat{\Phi} : G \times M \to M, \quad (g, p) \mapsto \Phi(g)(p) =: gp.$$

We only consider effective actions, that is, assume that the kernel of Φ is trivial.

The action of G on M is called *proper*, if the map

$$\Psi : G \times M \to M \times M, \quad (g, p) \mapsto (gp, p),$$

is proper, i.e. for every compact subset $C \subset M \times M$ its inverse image $\Psi^{-1}(C) \subset G \times M$ is compact as well. For example, the action is proper if G is compact. The properness of the action implies that: (i) G is locally compact, hence by [BM1], [BM2] (see also [MZ]) it carries the structure of a Lie group and the action map $\hat{\Phi}$ is smooth; (ii) Φ is a topological group

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isomorphism between G and $\Phi(G)$; (iii) $\Phi(G)$ is a closed subgroup of $\mathrm{Diff}(M)$ (see [Bi] for a brief survey on proper actions). Thus, one can assume that G is a Lie group acting smoothly and properly on the manifold M, and that it is realized as a closed subgroup of $\mathrm{Diff}(M)$.

Suppose now that M is equipped with a Riemannian metric $\mathcal{G}$, and let $\mathrm{Isom}(M,\mathcal{G})$ be the group of all isometries of M with respect to $\mathcal{G}$. It was shown in [MS] that $\mathrm{Isom}(M,\mathcal{G})$ acts properly on M (and so does its every closed subgroup). Conversely, by [Pal] (see also [Al]), for any Lie group acting properly on M there exists a C^∞-smooth G-invariant metric $\mathcal{G}$ on M. It then follows that Lie groups acting properly and effectively on the manifold M by diffeomorphisms are precisely closed subgroups of $\mathrm{Isom}(M,\mathcal{G})$ for all possible smooth Riemannian metrics $\mathcal{G}$ on M.

If G acts properly on M, then for every $p \in M$ its isotropy subgroup

$$G_p := \{g \in G : gp = p\}$$

is compact in G. Then by [Bo] the isotropy representation

$$\alpha_p : G_p \to GL(\mathbb{R}, T_p(M)), \quad g \mapsto dg(p) \tag{0.1}$$

is continuous and faithful, where $T_p(M)$ denotes the tangent space to M at p and $dg(p)$ is the differential of g at p. In particular, the linear isotropy subgroup

$$LG_p := \alpha_p(G_p)$$

is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to G_p. In some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the orthogonal group $O_m(\mathbb{R})$, where $m := \dim M$. Hence $\dim G_p \le \dim O_m(\mathbb{R}) = m(m-1)/2$. Furthermore, for every $p \in M$ its orbit

$$Gp := \{gp : g \in G\}$$

is a closed submanifold of M, and $\dim Gp \le m$. Thus, setting $d_G := \dim G$, we obtain

$$d_G = \dim G_p + \dim Gp \le m(m+1)/2.$$

It is a classical result (see [F], [C], [Ei]) that if G acts properly on a smooth manifold M of dimension $m \ge 2$ and $d_G = m(m+1)/2$, then M is isometric (with respect to some G-invariant metric) either to one of the standard complete simply-connected spaces of constant sectional curvature

$\mathbb{R}^m$, S^m, $\mathbb{H}^m$ (where $\mathbb{H}^m$ is the hyperbolic space), or to $\mathbb{RP}^m$. Next, it was shown in [Wa] (see also [Eg], [Y1]) that a group G with $m(m-1)/2+1 < d_G < m(m+1)/2$ cannot act properly on a smooth manifold M of dimension $m \neq 4$. The exceptional 4-dimensional case was considered in [Ish]; it turned out that a group of dimension 9 cannot act properly on a 4-dimensional manifold, and that if a 4-dimensional manifold admits a proper action of an 8-dimensional group G, then it has a G-invariant complex structure. Invariant complex structures will be discussed below in detail.

There exists also an explicit classification of pairs (M, G), where $m \geq 4$, G is connected, and $d_G = m(m-1)/2+1$ (see [Y1], [Ku], [O], [Ish]). Further, in [KN] a reasonably explicit classification of pairs (M, G), where $m \geq 6$, G is connected, and $(m-1)(m-2)/2+2 \leq d_G \leq m(m-1)/2$, was given. We also mention a classification of G-homogeneous manifolds for $m = 4$, $d_G = 6$ (see [Ish]) and a classifications of G-homogeneous simply-connected manifolds in the cases $m = 3$, $d_G = 3, 4$ and $m = 4$, $d_G = 5$ (see [C], [Pat]) obtained by E. Cartan's method of adapted frames introduced in [C]. There are many other results, especially for compact subgroups, but – to the best of our knowledge – no complete classifications exist beyond dimension $(m-1)(m-2)/2+2$ (see [Ko2], [Y2] and references therein for further details).

We study proper group actions in the complex setting with the general aim to build a theory for group dimensions lower than $(m-1)(m-2)/2+2$, thus extending – in this setting – the classical results mentioned above. In our setting real Lie groups act by holomorphic transformations on complex manifolds. Thus, from now on, M will denote a complex manifold of complex dimension n (hence $m = 2n$) and G will be a subgroup of $\text{Aut}(M)$, the group of all holomorphic automorphisms of M. We will be classifying pairs (M, G), but we will not be concerned with determining G-invariant Riemannian metrics on M.

Proper actions by holomorphic transformations are found in abundance. A fundamental result due to Kaup (see [Ka]) states that every closed subgroup of $\text{Aut}(M)$ that preserves a continuous distance on M acts properly on M. Thus, Lie groups acting properly and effectively on M by holomorphic transformations are precisely those closed subgroups of $\text{Aut}(M)$ that preserve continuous distances on M. In particular, if M is a Kobayashi-hyperbolic manifold, then $\text{Aut}(M)$ is a Lie group acting properly on M (see also [Ko1]).

In the complex setting, in some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the unitary group U_n. Hence $\dim G_p \leq \dim U_n = n^2$,

and therefore

$$d_G \le n^2 + 2n.$$

We note that $n^2 + 2n < (m-1)(m-2)/2 + 2$ for $m = 2n$ and $n \ge 5$. Thus, the group dimension range that arises in the complex case, for $n \ge 5$ lies strictly below the dimension range considered in the classical real case and therefore is not covered by the existing results. Furthermore, overlaps with these results for $n = 3, 4$ and $n = 2$, $d_G = 6$ occur only in relatively easy situations and do not lead to any significant simplifications in the complex case. The only interesting overlap with the real case occurs for $n = 2$, $d_G = 5$ (see [Pat]), but we do not discuss it in this paper. Note that in the situations when overlaps do occur, the existing classifications in the real case do not necessarily immediately lead to classifications in the complex case, since the determination of all G-invariant complex structures on the corresponding real manifolds may be a non-trivial task.

It was shown by Kaup in [Ka] that if $d_G = n^2 + 2n$, then M is holomorphically equivalent (in fact, holomorphically isometric with respect to some G-invariant metric) to one of $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$, $\mathbb{C}^n$, $\mathbb{CP}^n$, and an equivalence map F can be chosen so that the group $F \circ G \circ F^{-1} := \{F \circ g \circ F^{-1} : g \in G\}$ is, respectively, one of the groups $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$. Here $\mathrm{Aut}(\mathbb{B}^n) \simeq PSU_{n,1} := SU_{n,1}/(\text{center})$ is the group of all transformations

$$z \mapsto \frac{Az + b}{cz + d},$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n,1};$$

$G(\mathbb{C}^n) \simeq U_n \ltimes \mathbb{C}^n$ is the group of all holomorphic automorphisms of $\mathbb{C}^n$ of the form

$$z \mapsto Uz + a, \tag{0.2}$$

where $U \in U_n$, $a \in \mathbb{C}^n$ (we usually write $G(\mathbb{C})$ instead of $G(\mathbb{C}^1)$); and $G(\mathbb{CP}^n) \simeq PSU_{n+1} := SU_{n+1}/(\text{center})$ is the group of all holomorphic automorphisms of $\mathbb{CP}^n$ of the form

$$\zeta \mapsto U\zeta,$$

where ζ is a point in $\mathbb{CP}^n$ written in homogeneous coordinates, and $U \in SU_{n+1}$ (this group is a maximal compact subgroup of the complex Lie group

$\mathrm{Aut}(\mathbb{CP}^n) \simeq PSL_{n+1}(\mathbb{C}) := SL_{n+1}(\mathbb{C})/(\text{center}))$. In the above situation we say for brevity that F *transforms* G into one of $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$, respectively, and, in general, if $F : M_1 \to M_2$ is a biholomorphic map, $G_j \subset \mathrm{Aut}(M_j)$, $j = 1, 2$, are subgroups and $F \circ G_1 \circ F^{-1} = G_2$, we say that F transforms G_1 into G_2.

We remark that the groups $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$ are the full groups of holomorphic isometries of the Bergman metric on $\mathbb{B}^n$, the flat metric on $\mathbb{C}^n$, and the Fubini-Study metric on $\mathbb{CP}^n$, respectively, and that the above result due to Kaup can be obtained directly from the classification of Hermitian symmetric spaces (cf. [Ak], pp. 49–50).

We are interested in characterizing pairs (M, G) for $d_G < n^2 + 2n$, where $G \subset \mathrm{Aut}(M)$ acts on M properly. In [IKra], [I1], [I2], [I3] we considered the special case where M is a Kobayashi-hyperbolic manifold and $G = \mathrm{Aut}(M)$, and explicitly determined all manifolds with $n^2 - 1 \le d_{\mathrm{Aut}(M)} < n^2 + 2n$, $n \ge 2$ (see [I4] for a comprehensive exposition of these results). The case $d_{\mathrm{Aut}(M)} = n^2 - 2$ represents the first obstruction to the existence of an explicit classification, namely, there is no good description of hyperbolic manifolds with $n = 2$, $d_{\mathrm{Aut}(M)} = 2$ (see [I3], [I4]); it is possible, however, that a reasonable classification exists in this case for $n \ge 3$. Our immediate goal is to generalize these results to arbitrary proper actions on not necessarily Kobayashi-hyperbolic manifolds by classifying all pairs (M, G) with $n^2 - 1 \le d_G < n^2 + 2n$, $n \ge 2$, where G is assumed to be connected.

This classification problem splits into two cases: that of G-homogeneous manifolds and that of non-G-homogeneous ones (note that due to [Ka] G-homogeneity always takes place for $d_G > n^2$). While the techniques that we developed for non-homogeneous Kobayashi-hyperbolic manifolds seem to work well for general non-transitive proper actions, there is a substantial difference in the homogeneous case. Indeed, due to [N] every homogeneous Kobayashi-hyperbolic manifold is holomorphically equivalent to a Siegel domain of the second kind, and therefore such manifolds can be studied by using techniques available for Siegel domains (see e.g. [S]). This is how homogeneous Kobayashi-hyperbolic manifolds with $n^2 - 1 \le d_{\mathrm{Aut}(M)} < n^2 + 2n$, $n \ge 2$, were determined in [IKra], [I1], [I3], [I4]. This approach cannot be applied to general transitive proper actions, and one motivation for the present work is to re-obtain the classification of homogeneous Kobayashi-hyperbolic manifolds without using the non-trivial result of [N].

For general G-homogeneous manifolds we have

$$\dim G_p = d_G - 2n. \tag{0.3}$$

Hence for $n^2 - 1 \le d_G < n^2 + 2n$ we have $n^2 - 2n - 1 \le \dim G_p < n^2$. The starting point of our method of studying G-homogeneous manifolds with compact isotropy subgroups within the above dimension range is describing connected subgroups of the unitary group U_n of respective dimensions, thus determining the connected identity components of all possible linear isotropy subgroups. In the present paper we deal with manifolds equipped with proper actions for which $n^2 + 2 \le d_G < n^2 + 2n$. Due to [Ka], all such manifolds are G-homogeneous, and our proofs use the description of connected closed subgroups $H \subset U_n$ with $n^2 - 2n + 2 \le \dim H < n^2$ obtained in [IKra] (see also [I4]).

The first step towards a general classification for proper actions with $d_G < n^2+2n$ was in fact made in [IKra] where we observed that if $d_G \ge n^2+3$, then, as in the case $d_G = n^2+2n$, the manifold must be holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$. However, in [IKra] we did not investigate the question what groups (if any) are possible for each of these three manifolds within the dimension range $n^2 + 3 \le d_G < n^2 + 2n$. We resolve this question in Theorem 1.1 (see Section 1). Furthermore, in Theorem 2.1 we give a complete classification of all pairs (M, G) with $d_G = n^2 + 2$ (see Section 2).

In the proofs of Theorems 1.1 and 2.1 we do not use the existing structure theory for actions of Lie groups on complex manifolds (see e.g. [HO], [Wo]). Neither do we use the classification of Hermitian symmetric spaces due to E. Cartan (see [H]), a reference to which can significantly simplify the proof of Part (ii) of Theorem 1.1 and that of Theorem 2.1 (see Remark 2.2). We deliberately do not refer to these general facts and give proofs based on elementary calculations involving holomorphic fundamental vector fields of the G-action.

Working with lower values of d_G requires, in particular, further analysis of subgroups of U_n. For example, for the case $d_G = n^2+1$ one needs a description of closed connected $(n-1)^2$-dimensional subgroups. A description of such subgroups was given in Lemma 2.1 of [IKru], and we will attempt to deal with the case $d_G = n^2 + 1$ in our future work. There are a large number of examples of actions with $d_G = n^2+1$, and at this stage it is not clear whether all such actions can be classified in a reasonable way.

1 The case $n^2+3 \le d_G < n^2+2n$

In this section we prove the following theorem.

THEOREM 1.1 *Let M be a connected complex manifold of dimension $n \ge 2$ and $G \subset Aut(M)$ a connected Lie group that acts properly on M and has dimension d_G satisfying $n^2+3 \le d_G < n^2+2n$. Then one of the following holds:*

(i) M is holomorphically equivalent to $\mathbb{C}^n$ by means of a map that transforms G into the group $G_1(\mathbb{C}^n)$ which consists of all maps of the form (0.2) with $U \in SU_n$ (here $d_G = n^2+2n-1$);

(ii) $n=4$ and M is holomorphically equivalent to $\mathbb{C}^4$ by means of a map that transforms G into the group $G_2(\mathbb{C}^4)$ which consists of all maps of the form (0.2) for $n=4$ with $U \in e^{i\mathbb{R}}Sp_2$ (here $d_G = n^2+3 = 19$).[‡]

Proof: Fix $p \in M$. It follows from (0.3) that $n^2-2n+3 \le \dim LG_p < n^2$. Choose coordinates in $T_p(M)$ so that $LG_p \subset U_n$. Then Lemma 2.1 in [IKra] (see also Lemma 1.4 in [I4]) implies that the connected identity component LG_p^0 of LG_p either is SU_n or, for $n=4$, is conjugate in U_4 to $e^{i\mathbb{R}}Sp_2$. In both cases, it follows that LG_p acts transitively on directions in $T_p(M)$, that is, for any two non-zero vectors $v_1, v_2 \in T_p(M)$ there exists $h \in LG_p$ such that $hv_1 = \lambda v_2$ for some $\lambda \in \mathbb{R}^*$ (observe that the standard action of Sp_2 on $\mathbb{C}^4$ is transitive on the sphere $S^7 = \partial\mathbb{B}^4$). Now the result of [GK] gives that if M is non-compact, it is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, and an equivalence map can be chosen so that it maps p into the origin and transforms G_p into a subgroup of $U_n \subset G(\mathbb{C}^n)$; it then follows that one can find an equivalence map that transforms G_p^0 either into SU_n or, for $n=4$, into $e^{i\mathbb{R}}Sp_2$. Furthermore, the result of [BDK] gives that if M is compact, it is holomorphically equivalent to $\mathbb{CP}^n$.

Suppose first that $LG_p^0 = SU_n$. In this case $d_G = n^2+2n-1$. If M is holomorphically equivalent to $\mathbb{B}^n$, then the equivalence map transforms G into a closed subgroup of codimension 1 in $\mathrm{Aut}(\mathbb{B}^n)$. However, the Lie algebra of $\mathrm{Aut}(\mathbb{B}^n)$ is isomorphic to $\mathfrak{su}_{n,1}$, and it was shown in [EaI] that for $n \ge 2$ this algebra does not have codimension 1 subalgebras. Thus, M cannot be equivalent to $\mathbb{B}^n$. Next, if M is equivalent to $\mathbb{CP}^n$, the group G is compact.

[‡]Here Sp_2 denotes the standard compact real form of $Sp_4(\mathbb{C})$.

Therefore, the equivalence map transforms G into a closed codimension 1 subgroup of a maximal compact subgroup in $\mathrm{Aut}(\mathbb{CP}^n)$. It then follows that one can find an equivalence map that transforms G into a closed codimension 1 subgroup of $G(\mathbb{CP}^n)$. Since $G(\mathbb{CP}^n)$ is isomorphic to PSU_{n+1}, we obtain that SU_{n+1} has a closed codimension 1 subgroup, which contradicts Lemma 2.1 in [IKra] (see also Lemma 1.4 in [I4]). Thus, M cannot be equivalent to $\mathbb{CP}^n$ either.

Assume finally that M is equivalent to $\mathbb{C}^n$ and let F be an equivalence map that transforms G_p^0 into $SU_n \subset G(\mathbb{C}^n)$. We will show that F transforms G into $G_1(\mathbb{C}^n)$. We only give a proof for $n = 2$ (hence $d_G = 7$); the general case follows by considering copies of SU_2 lying in SU_n, and we omit details.

Denote by (z, w) coordinates in $\mathbb{C}^2$ and let $\mathfrak{g}$ be the Lie algebra of holomorphic vector fields on $\mathbb{C}^2$ that are fundamental vector fields of the action of $G^F := F \circ G \circ F^{-1}$, that is, $\mathfrak{g}$ consists of all vector fields X on $\mathbb{C}^2$ for which there exists an element a of the Lie algebra of G^F such that for all $(z, w) \in \mathbb{C}^2$ we have

$$X(z,w) = \frac{d}{dt}\Big[\exp(ta)(z,w)\Big]\Big|_{t=0}.$$

Since G^F acts on $\mathbb{C}^2$ transitively, the algebra $\mathfrak{g}$ is generated by $\mathfrak{su}_2$ (realized as the algebra of fundamental vector fields of the standard action of SU_2 on $\mathbb{C}^2$), and some vector fields

$$Y_j = f_j\,\partial/\partial z + g_j\,\partial/\partial w, \quad j = 1, 2, 3, 4.$$

Here the functions f_j, g_j, $j = 1, 2, 3, 4$, are holomorphic on $\mathbb{C}^2$ and satisfy the conditions

$$\begin{array}{ll} f_1(0) = 1, & g_1(0) = 0, \\ f_2(0) = i, & g_2(0) = 0, \\ f_3(0) = 0, & g_3(0) = 1, \\ f_4(0) = 0, & g_4(0) = i. \end{array}$$

To prove that $G^F = G_1(\mathbb{C}^n)$, it is sufficient to show that Y_j can be chosen as follows:

$$\begin{array}{rcl} Y_1 & = & \partial/\partial z, \\ Y_2 & = & i\,\partial/\partial z, \\ Y_3 & = & \partial/\partial w, \\ Y_4 & = & i\,\partial/\partial w. \end{array} \tag{1.1}$$

We fix the following generators in $\mathfrak{su}_2$:

$$\begin{array}{rcrcr} X_1 & := & iz\,\partial/\partial z & - & iw\,\partial/\partial w, \\ X_2 & := & w\,\partial/\partial z & - & z\,\partial/\partial w, \\ X_3 & := & iw\,\partial/\partial z & + & iz\,\partial/\partial w. \end{array}$$

A straightforward calculation gives

$$\begin{array}{rcr} [Y_1, X_1](0) & = & (i, 0), \\ [Y_2, X_1](0) & = & -(1, 0). \end{array}$$

It then follows that

$$\begin{array}{rcrl} Y_1 & = & -[Y_2, X_1] & (\mathrm{mod}\ \mathfrak{su}_2), \\ Y_2 & = & [Y_1, X_1] & (\mathrm{mod}\ \mathfrak{su}_2), \end{array} \tag{1.2}$$

which implies

$$Y_1 = -[[Y_1, X_1], X_1] \ \ (\mathrm{mod}\ \mathfrak{su}_2).$$

This identity yields

$$\begin{aligned} &\Bigg(z\,\frac{\partial f_1}{\partial z} - 3w\,\frac{\partial f_1}{\partial w} - z^2\,\frac{\partial^2 f_1}{\partial z^2} + \\ &\qquad\qquad 2zw\,\frac{\partial^2 f_1}{\partial z\,\partial w} - w^2\,\frac{\partial^2 f_1}{\partial w^2}\Bigg)\,\partial/\partial z + \\ &\Bigg(-3z\,\frac{\partial g_1}{\partial z} + w\,\frac{\partial g_1}{\partial w} - z^2\,\frac{\partial^2 g_1}{\partial z^2} + \\ &\qquad\qquad 2zw\,\frac{\partial^2 g_1}{\partial z\,\partial w} - w^2\,\frac{\partial^2 g_1}{\partial w^2}\Bigg)\,\partial/\partial w = 0 \ \ (\mathrm{mod}\ \mathfrak{su}_2). \end{aligned} \tag{1.3}$$

Representing the functions f_1 and g_1 as power series around the origin, plugging these representations into (1.3) and collecting terms of fixed orders, we obtain

$$\begin{aligned} Y_1 = &\left(1 + \sum_{m=1}^{\infty}\left(\alpha_m z^m w^m + \alpha_m' z^{m+1} w^{m-1}\right)\right)\,\partial/\partial z + \\ &\left(\sum_{m=1}^{\infty}\left(\beta_m z^m w^m + \beta_m' z^{m-1} w^{m+1}\right)\right)\,\partial/\partial w \ \ (\mathrm{mod}\ \mathfrak{su}_2), \end{aligned}$$

for some $\alpha_m, \alpha'_m, \beta_m, \beta'_m \in \mathbb{C}$. Adding to Y_1 an element of $\mathfrak{su}_2$ if necessary, we can assume that Y_1 has no linear terms, that is

$$\begin{aligned} Y_1 = & \left(1 + \sum_{m=1}^{\infty} \left(\alpha_m z^m w^m + \alpha'_m z^{m+1} w^{m-1}\right)\right) \partial/\partial z + \\ & \left(\sum_{m=1}^{\infty} \left(\beta_m z^m w^m + \beta'_m z^{m-1} w^{m+1}\right)\right) \partial/\partial w. \end{aligned} \tag{1.4}$$

Further, (1.2) gives

$$Y_2 = -[[Y_2, X_1], X_1] \pmod{\mathfrak{su}_2},$$

and the application of an analogous argument to Y_2 yields that Y_2 can be chosen to have the form

$$\begin{aligned} Y_2 = & \left(i + \sum_{m=1}^{\infty} \left(\tilde{\alpha}_m z^m w^m + \tilde{\alpha}'_m z^{m+1} w^{m-1}\right)\right) \partial/\partial z + \\ & \left(\sum_{m=1}^{\infty} \left(\tilde{\beta}_m z^m w^m + \tilde{\beta}'_m z^{m-1} w^{m+1}\right)\right) \partial/\partial w. \end{aligned} \tag{1.5}$$

for some $\tilde{\alpha}_m, \tilde{\alpha}'_m, \tilde{\beta}_m, \tilde{\beta}'_m \in \mathbb{C}$. Next, plugging (1.4), (1.5) into either of identities (1.2) implies

$$\begin{array}{ll} \tilde{\alpha}_m = i\alpha_m, & \tilde{\alpha}'_m = -i\alpha'_m, \\ \tilde{\beta}_m = -i\beta_m, & \tilde{\beta}'_m = i\beta'_m, \end{array}$$

for all $m \in \mathbb{N}$. We also observe that considering $[Y_3, X_1]$ and $[Y_4, X_1]$ yields that Y_3, Y_4 can be chosen to have the forms

$$\begin{aligned} Y_3 = & \left(\sum_{m=1}^{\infty} \left(\gamma_m z^m w^m + \gamma'_m z^{m+1} w^{m-1}\right)\right) \partial/\partial z + \\ & \left(1 + \sum_{m=1}^{\infty} \left(\delta_m z^m w^m + \delta'_m z^{m-1} w^{m+1}\right)\right) \partial/\partial w, \end{aligned}$$

$$\begin{aligned} Y_4 = & \left(\sum_{m=1}^{\infty} \left(-i\gamma_m z^m w^m + i\gamma'_m z^{m+1} w^{m-1}\right)\right) \partial/\partial z + \\ & \left(i + \sum_{m=1}^{\infty} \left(i\delta_m z^m w^m - i\delta'_m z^{m-1} w^{m+1}\right)\right) \partial/\partial w, \end{aligned}$$

for some $\gamma_m, \gamma'_m, \delta_m, \delta'_m \in \mathbb{C}$.

Further, computing $[Y_j, X_2]$ for $j = 1, 2, 3, 4$ and collecting terms of orders 2 and greater, we obtain

$$\begin{aligned}
&\alpha_m = \alpha'_m = \beta_m = \beta'_m = \gamma_m = \gamma'_m = \delta_m = \delta'_m = 0, \quad m \geq 2,\\
&\alpha'_1 = \beta_1,\ \beta'_1 = \alpha_1,\ \gamma'_1 = \delta_1,\ \delta'_1 = \gamma_1.
\end{aligned}$$

Next, we have

$$[Y_1, Y_2] = -2i\beta_1(2z\,\partial/\partial z + w\,\partial/\partial w).$$

Hence $[Y_1, Y_2] = 0\,(\mathrm{mod}\ \mathfrak{su}_2)$, which can only hold if $\beta_1 = 0$. Similarly, considering $[Y_3, Y_4]$ leads to $\gamma_1 = 0$. Finally, we compute $[Y_1, Y_3]$, $[Y_1, Y_4]$ and see that $\alpha_1 = \delta_1 = 0$. Thus, Y_j chosen as above (that is, not having linear terms) are in fact given by (1.1), and we have obtained (i) of the theorem.

Suppose next that $n = 4$ and LG_p^0 is conjugate in U_4 to $e^{i\mathbb{R}}Sp_2$. In this case $d_G = n^2 + 3 = 19$. If M is equivalent to $\mathbb{CP}^4$, the group G is compact. Therefore, one can find an equivalence map that transforms G into a closed 19-dimensional subgroup of $G(\mathbb{CP}^4)$. Since $G(\mathbb{CP}^4)$ is isomorphic to PSU_5, we obtain that SU_5 has a closed 19-dimensional subgroup, which contradicts Lemma 2.1 in [IKra] (see also Lemma 1.4 in [I4]). Thus, M cannot be equivalent to $\mathbb{CP}^4$.

Assume now that $n = 4$, the manifold M is equivalent to one of $\mathbb{B}^4$, $\mathbb{C}^4$, and let F be an equivalence map that transforms G_p^0 into $e^{i\mathbb{R}}Sp_2 \subset G(\mathbb{C}^4)$. We will show that F transforms G into $G_2(\mathbb{C}^4)$. Let $\mathfrak{g}$ be the Lie algebra of fundamental vector fields of the action of $G^F := F \circ G \circ F^{-1}$ on one of $\mathbb{B}^4$, $\mathbb{C}^4$, respectively. Since G^F contains the one-parameter subgroup $z \mapsto e^{it}z$, $t \in \mathbb{R}$, the algebra $\mathfrak{g}$ contains the vector field

$$Z_0 := i\sum_{k=1}^{4} z_k\,\partial/\partial z_k.$$

Hilfssatz 4.8 of [Ka] then gives that every vector field in $\mathfrak{g}$ is polynomial and has degree at most 2. Since G^F acts transitively on one of $\mathbb{B}^4$, $\mathbb{C}^4$, the algebra $\mathfrak{g}$ is generated by $\langle Z_0\rangle \oplus \mathfrak{sp}_2$ (where $\langle Z_0\rangle$ is the one-dimensional algebra spanned by Z_0 and $\mathfrak{sp}_2$ denotes the Lie algebra of Sp_2 realized as the algebra of fundamental vector fields of the standard action of Sp_2 on $\mathbb{C}^4$),

and some vector fields

$$\begin{aligned} V_j &= \sum_{k=1}^{4} f_j^k\,\partial/\partial z_k, \\ W_j &= \sum_{k=1}^{4} g_j^k\,\partial/\partial z_k, \end{aligned} \tag{1.6}$$

for $j = 1,2,3,4$. Here f_j^k, g_j^k are holomorphic polynomials of degree at most 2 such that

$$f_j^k(0) = \delta_j^k, \quad g_j^k(0) = i\delta_j^k. \tag{1.7}$$

Considering $[Z_0,[V_j,Z_0]]$, $[Z_0,[W_j,Z_0]]$ instead of V_j, W_j if necessary, we can assume that V_j, W_j, $j = 1,2,3,4$, have no linear terms (see the proof of Satz 4.9 in [Ka]). Thus, we have

$$f_j^k = \delta_j^k + \text{second-order terms}, \quad g_j^k = i\delta_j^k + \text{second-order terms}.$$

To prove that F maps M onto $\mathbb{C}^4$ and $G^F = G_2(\mathbb{C}^4)$, we need to show that all the second-order terms identically vanish, that is,

$$f_j^k \equiv \delta_j^k, \quad g_j^k \equiv i\delta_j^k. \tag{1.8}$$

We introduce the following vector fields from $\langle Z_0\rangle \oplus \mathfrak{sp}_2$:

$$\begin{aligned} Z_1 &:= iz_1\,\partial/\partial z_1 + iz_4\,\partial/\partial z_4, \\ Z_2 &:= iz_2\,\partial/\partial z_2 - iz_4\,\partial/\partial z_4, \\ Z_3 &:= iz_3\,\partial/\partial z_3 + iz_4\,\partial/\partial z_4, \\ Z_4 &:= z_4\,\partial/\partial z_2 - z_2\,\partial/\partial z_4, \\ Z_5 &:= iz_4\,\partial/\partial z_2 + iz_2\,\partial/\partial z_4, \\ Z_6 &:= z_2\,\partial/\partial z_1 - z_1\,\partial/\partial z_2 + z_4\,\partial/\partial z_3 - z_3\,\partial/\partial z_4, \\ Z_7 &:= iz_2\,\partial/\partial z_1 + iz_1\,\partial/\partial z_2 - iz_4\,\partial/\partial z_3 - iz_3\,\partial/\partial z_4, \\ Z_8 &:= z_3\,\partial/\partial z_1 - z_1\,\partial/\partial z_3, \\ Z_9 &:= iz_3\,\partial/\partial z_1 + iz_1\,\partial/\partial z_3. \end{aligned} \tag{1.9}$$

It is straightforward to see that the commutators $[V_1,Z_2]$ and $[V_1,Z_3]$ vanish at the origin and have no linear terms. Hence these commutators are equal to 0, which implies that V_1 has the form

$$V_1 = (1+\alpha z_1^2)\,\partial/\partial z_1 + \beta z_1 z_2\,\partial/\partial z_2 + (\gamma z_1 z_3 + \delta z_2 z_4)\,\partial/\partial z_3 + \varepsilon z_1 z_4\,\partial/\partial z_4, \tag{1.10}$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{C}$. Similarly, considering $[W_1, Z_2]$ and $[W_1, Z_3]$ gives

$$W_1 = (i + \alpha' z_1^2)\, \partial/\partial z_1 + \beta' z_1 z_2\, \partial/\partial z_2 + (\gamma' z_1 z_3 + \delta' z_2 z_4)\, \partial/\partial z_3 + \varepsilon' z_1 z_4\, \partial/\partial z_4, \tag{1.11}$$

for some $\alpha', \beta', \gamma', \delta', \varepsilon' \in \mathbb{C}$.

Consider the commutator $[V_1, Z_1]$. It is straightforward to see from (1.10) that $[V_1, Z_1]$ does not have linear terms and that $[V_1, Z_1] - W_1$ vanishes at the origin. Hence $[V_1, Z_1] = W_1$ which implies that in (1.11) we have

$$\alpha' = -i\alpha,\ \beta' = -i\beta,\ \gamma' = -i\gamma,\ \delta' = -i\delta,\ \varepsilon' = -i\varepsilon.$$

Then

$$[V_1, W_1] = -2i\, (2\alpha z_1\, \partial/\partial z_1 + \beta z_2\, \partial/\partial z_2 + \gamma z_3\, \partial/\partial z_3 + \varepsilon z_4\, \partial/\partial z_4)\, .$$

Therefore, $[V_1, W_1] = 0\, (\mathrm{mod}\ \langle Z_0 \rangle \oplus \mathfrak{sp}_2)$, which can only hold if

$$\varepsilon = 2\alpha - \beta + \gamma. \tag{1.12}$$

Next, we compute

$$[V_1, Z_4] = (\varepsilon - \beta) z_1 z_4\, \partial/\partial z_2 + \delta(z_2^2 - z_4^2)\, \partial/\partial z_3 + (\varepsilon - \beta) z_1 z_2\, \partial/\partial z_4.$$

Since $[V_1, Z_4]$ does not have linear terms and vanishes at the origin, it vanishes identically, that is, we have

$$\varepsilon = \beta,\ \delta = 0. \tag{1.13}$$

Further, computing the commutators $[V_2, Z_1]$ and $[V_2, Z_3]$, we obtain analogously to (1.10)

$$V_2 = \kappa z_1 z_2\, \partial/\partial z_1 + (1 + \lambda z_2^2)\, \partial/\partial z_2 + \mu z_2 z_3\, \partial/\partial z_3 + (\nu z_1 z_3 + \xi z_2 z_4)\, \partial/\partial z_4, \tag{1.14}$$

for some $\kappa, \lambda, \mu, \nu, \xi \in \mathbb{C}$. In addition, it is straightforward to see that $[V_1, Z_6]$ does not have linear terms and that $[V_1, Z_6] + V_2$ vanishes at the origin. Hence we have

$$[V_1, Z_6] = -V_2. \tag{1.15}$$

Formulas (1.14) and (1.15) imply

$$\beta = \alpha,\ \varepsilon = \gamma - \delta. \tag{1.16}$$

Now (1.12), (1.13), (1.16) yield

$$\alpha=\beta=\gamma=\delta=\varepsilon=0,$$

and therefore

$$V_1=\partial/\partial z_1,\ W_1=i\,\partial/\partial z_1.$$

It then follows from (1.15) that

$$V_2=\partial/\partial z_2.$$

Next, we have

$$\begin{array}{rcl}[V_1,Z_7] & = & i\,\partial/\partial z_2,\\ [V_1,Z_8] & = & -\,\partial/\partial z_3,\\ [V_1,Z_9] & = & i\,\partial/\partial z_3,\\ [V_2,Z_4] & = & -\,\partial/\partial z_4,\\ [V_2,Z_5] & = & i\,\partial/\partial z_4,\end{array}\tag{1.17}$$

hence the vector fields in the right-hand side of formulas (1.17) lie in $\mathfrak{g}$. Since $W_2-i\,\partial/\partial z_2$, $V_3-\partial/\partial z_3$, $W_3-i\,\partial/\partial z_3$, $V_4-\partial/\partial z_4$, $W_4-i\,\partial/\partial z_4$ have no linear terms and vanish at the origin, they vanish identically and we obtain

$$\begin{array}{rcl}W_2 & = & i\,\partial/\partial z_2,\\ V_3 & = & \partial/\partial z_3,\\ W_3 & = & i\,\partial/\partial z_3,\\ V_4 & = & \partial/\partial z_4,\\ W_4 & = & i\,\partial/\partial z_4.\end{array}\tag{1.18}$$

Thus, (1.8) holds, and we have obtained (ii) of the theorem.

The proof is complete. □

2 The case $d_G=n^2+2$

In this section we obtain the following result.

THEOREM 2.1 *Let M be a connected complex manifold of dimension $n\geq 2$ and $G\subset Aut(M)$ a connected Lie group that acts properly on M and has dimension $d_G=n^2+2$. Then one of the following holds:*

(i) M is holomorphically equivalent to $M'\times M''$, where M' is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, and M'' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$; an equivalence map can be chosen

so that it transforms G *into* $G' \times G''$, *where* G' *is one of* $Aut(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, *and* G'' *is one of* $Aut(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, *respectively;*

(ii) $n = 4$ *and* M *is holomorphically equivalent to* $\mathbb{C}^4$ *by means of a map that transforms* G *into the group* $G_3(\mathbb{C}^4)$ *which consists of all maps of the form (0.2) for* $n = 4$ *with* $U \in Sp_2$.

Proof: Fix $p \in M$. It follows from (0.3) that $\dim LG_p = n^2 - 2n + 2$. Choose coordinates in $T_p(M)$ so that $LG_p \subset U_n$. Then Lemma 2.1 in [IKra] implies that the connected identity component LG_p^0 of LG_p either is conjugate in U_n to $U_{n-1} \times U_1$ or, for $n = 4$, is conjugate in U_4 to Sp_2.

Suppose first that LG_p^0 is conjugate to $U_{n-1} \times U_1$. By Bochner's linearization theorem (see [Bo]) there exist a G_p-invariant neighborhood $\mathcal{V}$ of p in M, an LG_p-invariant neighborhood $\mathcal{U}$ of the origin in $T_p(M)$ and a biholomorphic map $F : \mathcal{V} \to \mathcal{U}$, with $F(p) = 0$, such that for every $g \in G_p$ the following holds in $\mathcal{V}$:

$$F \circ g = \alpha_p(g) \circ F,$$

where α_p is the isotropy representation at p (see (0.1)). Let $\mathfrak{g}_M$ be the Lie algebra of fundamental vector fields of the action of G on M, and $\mathfrak{g}_{\mathcal{V}}$ the Lie algebra of the restrictions of the elements of $\mathfrak{g}_M$ to $\mathcal{V}$. Denote by $\mathfrak{g}$ the Lie algebra of vector fields on $\mathcal{U}$ obtained by pushing forward the elements of $\mathfrak{g}_{\mathcal{V}}$ by means of F. Observe that $\mathfrak{g}_M$, $\mathfrak{g}_{\mathcal{V}}$, $\mathfrak{g}$ are naturally isomorphic, and we denote by $\varphi : \mathfrak{g}_M \to \mathfrak{g}$ the isomorphism induced by F.

Choose coordinates $(z_1, \dots, z_n)$ in $T_p(M)$ so that in these coordinates LG_p^0 is the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \tag{2.1}$$

where $A \in U_{n-1}$, $\alpha \in \mathbb{R}$. Since F transforms $G_p^0|_{\mathcal{V}}$ into $LG_p^0|_{\mathcal{U}}$ and since G acts transitively on M, the algebra $\mathfrak{g}$ is generated by $\mathfrak{u}_{n-1} \oplus \mathfrak{u}_1$ and some vector fields

$$V_j = \sum_{k=1}^{n} f_j^k \, \partial/\partial z_k,$$

$$W_j = \sum_{k=1}^{n} g_j^k \, \partial/\partial z_k,$$

for $j=1,\dots,n$, where f_j^k, g_j^k are holomorphic functions on $\mathcal{U}$ such that

$$f_j^k(0)=\delta_j^k,\quad g_j^k(0)=i\delta_j^k.$$

Here $\mathfrak{u}_{n-1}\oplus\mathfrak{u}_1$ is realized as the algebra of vector fields on $\mathcal{U}$ of the form

$$\sum_{j=1}^{n-1}(a_{j\,1}z_1+\ldots+a_{j\,n-1}z_{n-1})\,\partial/\partial z_j+iaz_n\,\partial/\partial z_n, \tag{2.2}$$

where

$$\begin{pmatrix} a_{1\,1} & \dots & a_{1\,n-1}\\ \vdots & \vdots & \vdots\\ a_{n-1\,1} & \dots & a_{n-1\,n-1}\end{pmatrix}\in\mathfrak{u}_{n-1},$$

and $a\in\mathbb{R}$.

Observe that $\mathfrak{g}$ contains the vector field

$$Z_0:=i\sum_{k=1}^{n}z_k\,\partial/\partial z_k.$$

Therefore, due to Hilfssatz 4.8 of [Ka], every vector field in $\mathfrak{g}$ is polynomial and has degree at most 2. Next, considering $[Z_0,[V_j,Z_0]]$, $[Z_0,[W_j,Z_0]]$ instead of V_j, W_j if necessary, we can assume that V_j, W_j, $j=1,\dots,n$, have no linear terms.

Furthermore, $\mathfrak{g}$ contains the vector fields

$$Z_k:=iz_k\,\partial/\partial z_k,\quad k=1,\dots,n.$$

Since for each $j=1,\dots,n$ the commutators $[V_j,Z_k]$ and $[W_j,Z_k]$, with $k\neq j$, vanish at the origin and do not contain linear terms, they vanish identically, which gives

$$\begin{aligned} V_j &= \sum_{k\neq j}\alpha_j^k z_kz_j\,\partial/\partial z_k+(1+\alpha_j^j z_j^2)\,\partial/\partial z_j,\\ W_j &= \sum_{k\neq j}\beta_j^k z_kz_j\,\partial/\partial z_k+(i+\beta_j^j z_j^2)\,\partial/\partial z_j,\end{aligned}$$

for some $\alpha_j^k,\beta_j^k\in\mathbb{C}$ (cf. the proof of Proposition 4.9 of [Ka]). Next, $[V_j,Z_j]$ has no linear terms and $[V_j,Z_j](0)=(0,\dots,0,i,0,\dots,0)$, where i occurs in the jth position. Hence $[V_j,Z_j]=W_j$ which implies

$$\beta_j^k=-i\alpha_j^k,$$

for all j, k. Now, for $j = 1, \dots, n-1$ consider the commutator $[V_j, V_n]$. Clearly, the linear part $\mathcal{L}_j$ of this commutator must be an element of $\mathfrak{u}_{n-1} \oplus \mathfrak{u}_1$. It is straightforward to see that

$$\mathcal{L}_j = \alpha_n^j z_n \, \partial/\partial z_j - \alpha_j^n z_j \, \partial/\partial z_n,$$

which can lie in $\mathfrak{u}_{n-1} \oplus \mathfrak{u}_1$ only if $\alpha_j^n = \alpha_n^j = 0$ (see (2.2)).

Thus, we have shown that for $j = 1, \dots, n-1$ the vector fields V_j, W_j do not depend on z_n and the vector fields V_n, W_n do not depend on z_j. Accordingly, we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ is the ideal generated by $\mathfrak{u}_{n-1}$ and V_j, W_j, for $j = 1, \dots, n-1$, and $\mathfrak{g}_2$ is the ideal generated by $\mathfrak{u}_1$ and V_n, W_n.

Let G_j be the connected normal (possibly non-closed) subgroup of G with Lie algebra $\tilde{\mathfrak{g}}_j := \varphi^{-1}(\mathfrak{g}_j) \subset \mathfrak{g}_M$ for $j = 1, 2$. Clearly, for each j the subgroup G_j contains $\alpha_p^{-1}(L_{j\,p}) \subset G_p^0$, where $L_{1\,p} \simeq U_{n-1}$ and $L_{2\,p} \simeq U_1$ are the subgroups of LG_p^0 given by $\alpha = 0$ and $A = \mathrm{id}$ in formula (2.1), respectively. Consider the orbit $G_j p$, $j = 1, 2$. Clearly, for each j there exists a neighborhood $\mathcal{W}_j$ of the identity in G_j such that

$$\begin{aligned} \mathcal{W}_1 p &= F^{-1}\left(\mathcal{U}' \cap \{z_n = 0\}\right), \\ \mathcal{W}_2 p &= F^{-1}\left(\mathcal{U}' \cap \{z_1 = 0, \dots, z_{n-1} = 0\}\right), \end{aligned}$$

for some neighborhood $\mathcal{U}' \subset \mathcal{U}$ of the origin in $T_p(M)$. Thus, each $G_j p$ is a complex (possibly non-closed) submanifold of M, and the ideal $\tilde{\mathfrak{g}}_j$ consists exactly of those vector fields from $\mathfrak{g}_M$ that are tangent to $G_j p$ at some point (and hence at all points).

Furthermore, for the isotropy subgroup $G_{j\,p}$ of the point p with respect to the G_j-action we have $G_{j\,p}^0 = \alpha_p^{-1}(L_{j\,p})$, $j = 1, 2$. Since $L_{j\,p}$ acts transitively on real directions in $T_p(G_j p)$ for $j = 1, 2$, by [GK], [BDK] we obtain that $G_1 p$ is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ and $G_2 p$ is holomorphically equivalent to one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$.

We will now show that each G_j is closed in G. We assume that $j = 1$; for $j = 2$ the proof is identical. Let $\mathfrak{U}$ be a neighborhood of 0 in $\mathfrak{g}_M$ where the exponential map into G is a diffeomorphism, and let $\mathfrak{V} := \exp(\mathfrak{U})$. To prove that G_1 is closed in G it is sufficient to show that for some neighborhood $\mathfrak{W}$ of $e \in G$, $\mathfrak{W} \subset \mathfrak{V}$, we have $G_1 \cap \mathfrak{W} = \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}) \cap \mathfrak{W}$. Assuming the opposite we obtain a sequence $\{g_j\}$ of elements of G_1 converging to e in G such that for every j we have $g_j = \exp(a_j)$ with $a_j \in \mathfrak{U} \setminus \tilde{\mathfrak{g}}_1$. Observe now that there exists a neighborhood $\mathcal{V}'$ of p in M foliated by complex

submanifolds holomorphically equivalent to $\mathbb{B}^{n-1}$ in such a way that the leaf passing through p lies in G_1p. Specifically, we take $\mathcal{V}' := F^{-1}(\mathcal{U}')$ for a suitable neighborhood $\mathcal{U}' \subset \mathcal{U}$ of the origin in $T_p(M)$, and the leaves of the foliation are then given as $F^{-1}(\mathcal{U}' \cap \{z_n = \text{const}\})$. For every $s \in \mathcal{V}'$ we denote by N_s the leaf of the foliation passing through s. Observe that for every $s \in \mathcal{V}'$ vector fields from $\tilde{\mathfrak{g}}_1$ are tangent to N_s at every point. Let $p_j := g_jp$. If j is sufficiently large, we have $p_j \in \mathcal{V}'$. We will now show that $N_{p_j} \neq N_p$ for large j.

Let $\mathfrak{U}'' \subset \mathfrak{U}' \subset \mathfrak{U}$ be neighborhoods of 0 in $\mathfrak{g}_M$ such that: (a) $\exp(\mathfrak{U}'') \cdot \exp(\mathfrak{U}'') \subset \exp(\mathfrak{U}')$; (b) $\exp(\mathfrak{U}'') \cdot \exp(\mathfrak{U}') \subset \exp(\mathfrak{U})$; (c) $\mathfrak{U}' = -\mathfrak{U}'$; (d) $G_{1\,p} \cap \exp(\mathfrak{U}') \subset \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}')$. We also assume that $\mathcal{V}'$ is chosen so that $N_p \subset \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}'')p$. Suppose that $p_j \in N_p$. Then $p_j = sp$ for some $s \in \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}'')$ and hence $t := g_j^{-1}s$ is an element of $G_{1\,p}$. For large j we have $g_j^{-1} \in \exp(\mathfrak{U}'')$. Condition (a) now implies that $t \in \exp(\mathfrak{U}')$ and hence by (c), (d) we have $t^{-1} \in \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}')$. Therefore, by (b) we obtain $g_j \in \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U})$ which contradicts our choice of g_j. Thus, for large j the leaves N_{p_j} are distinct from N_p. Furthermore, they accumulate to $N_p \subset G_1p$. At the same time, since vector fields from $\tilde{\mathfrak{g}}_1$ are tangent to every N_{p_j}, we have $N_{p_j} \subset G_1p$ for all j, and thus the orbit G_1p accumulates to itself. Below we will show that this is in fact impossible thus obtaining a contradiction. Clearly, we only need to consider the case when G_1p is non-compact, that is, equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$.

Since $G^0_{1\,p}$ acts on G_1p effectively, by the result of [GK], the orbit G_1p is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$ by means of a map that takes p into the origin and transforms $G^0_{1\,p}$ into $U_{n-1} \subset G(\mathbb{C}^{n-1})$. Consider the set $S := G_1p \cap G_2p$. The orbit G_1p accumulates to itself, and therefore S contains a point other than p. Note that S does not contain any curve. Since $G^0_{1\,p}$ preserves each of G_1p, G_2p, it preserves S. However, the $G^0_{1\,p}$-orbit of every point in G_1p other than p is a hypersurface in G_1p diffeomorphic to the sphere S^{2n-3}. This contradiction shows that in fact S consists of p alone, and hence G_1 is closed in G.

Thus, G_j is closed in G for $j = 1, 2$. Hence G_j acts on M properly and G_jp is a closed submanifold of M for each j. Recall that G_1p is equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ and G_2p is equivalent to one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, and denote by F_1, F_2 the respective equivalence maps. Let $K_j \subset G_j$ be the ineffectivity kernel of the G_j-action on G_jp for $j = 1, 2$. Clearly, $K_j \subset G_{j\,p}$ and, since $G^0_{j\,p}$ acts on G_jp effectively, K_j is a discrete normal subgroup of G_j for each j (in particular, K_j lies in the center of G_j for $j = 1, 2$). Since

$d_{G_1} = n^2 - 1 = (n-1)^2 + 2(n-1)$ and $d_{G_2} = 3$, the results of [Ka] yield that F_1 can be chosen to transform G_1/K_1 into one of $\mathrm{Aut}(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, respectively, and F_2 can be chosen to transform G_2/K_2 into one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively, where G_j/K_j is viewed as a subgroup of $\mathrm{Aut}(G_j p)$ for each j.

We will now show that the subgroup K_j is in fact trivial for each $j = 1, 2$. We only consider the case $j = 1$ since for $j = 2$ the proof is identical. Clearly, $K_1 \setminus \{e\} \subset G_{1\,p} \setminus G^0_{1\,p}$, and if K_1 is non-trivial, the compact group $G_{1\,p}$ is disconnected. Observe that any maximal compact subgroup of each of $\mathrm{Aut}(\mathbb{B}^{n-1}) \simeq PSU_{n-1,1}$ and $G(\mathbb{C}^{n-1}) \simeq U_{n-1} \ltimes \mathbb{C}^{n-1}$ is isomorphic to U_{n-1} and therefore, if G_1/K_1 is isomorphic to either of these two groups, it follows that $G_{1\,p}$ is a maximal compact subgroup of G_1. Since G_1 is connected, so is $G_{1\,p}$, and therefore K_1 is trivial in either of these two cases. Suppose now that G_1/K_1 is isomorphic to $G(\mathbb{CP}^{n-1}) \simeq PSU_n$. Then the universal cover of G_1 is the group SU_n, and let $\Pi : SU_n \to G_1$ be a covering homomorphism. Then $\Pi^{-1}(G^0_{1\,p})^0$ is a closed $(n-1)^2$-dimensional connected subgroup of SU_n. It follows from Lemma 2.1 of [IKru] that $\Pi^{-1}(G^0_{1\,p})^0$ is conjugate in SU_n to the subgroup of matrices of the form

$$\begin{pmatrix} 1/\det B & 0 \\ 0 & B \end{pmatrix}, \tag{2.3}$$

where $B \in U_{n-1}$. This yields that $\Pi^{-1}(G^0_{1\,p})^0$ contains the center of SU_n, hence $G^0_{1\,p}$ contains the center of G_1. In particular, $K_1 \subset G^0_{1\,p}$ which implies that K_1 is trivial in this case as well. Thus, G_1 is isomorphic to one of $\mathrm{Aut}(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$ and G_2 is isomorphic to one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$.

We remark here that since M is G-homogeneous and G_j is normal in G, the discussion above remains valid for any point $q \in M$ in place of p; in particular, all G_j-orbits are pairwise holomorphically equivalent, for $j = 1, 2$.

Next, since $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the group G is a locally direct product of G_1 and G_2. We claim that $H := G_1 \cap G_2$ is trivial. Indeed, H is a discrete normal subgroup of each of G_1, G_2. However, every discrete normal subgroup of each of $\mathrm{Aut}(\mathbb{B}^k)$, $G(\mathbb{C}^k)$, $G(\mathbb{CP}^k)$ for $k \in \mathbb{N}$ is trivial, since the center of each of these groups is trivial. Hence H is trivial and therefore $G = G_1 \times G_2$.

We will now show that for every $q_1, q_2 \in M$ the orbits $G_1 q_1$ and $G_2 q_2$ intersect at exactly one point. Let $g \in G$ be an element such that $g q_2 = q_1$. It can be uniquely represented in the form $g = g_1 g_2$ with $g_j \in G_j$ for $j = 1, 2$,

and therefore we have $g_2q_2 = g_1^{-1}q_1$. Hence the intersection $G_1q_1 \cap G_2q_2$ is non-empty. We will now prove that $G_1q \cap G_2q = \{q\}$ for every $q \in M$. Suppose that for some $q \in M$ the intersection $G_1q \cap G_2q$ contain a point $q' \neq q$. Let $g_1 \in G_1$ be an element such that $g_1q = q'$. Clearly, g_1 preserves G_2q. Since $g_1 \in G_1$ and $G = G_1 \times G_2$, the element g_1 commutes with every element of G_2. Consider the restriction $g_1' := g_1|_{G_2q}$. Let $\hat{F}$ be a biholomorphic map from G_2q onto one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$ that transforms G_2 into one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively. Then $\hat{F}$ transforms g_1' into a holomorphic automorphism of one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$ that lies in the centralizer of the corresponding group. In each of the three cases we immediately see that g_1' is the identity, which is a contradiction. Thus, the intersection $G_1q \cap G_2q$ consists of q alone for every $q \in M$.

Let, as before, F_1 be a biholomorphic map from G_1p onto M', where M' is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, that transforms G_1 into G', where G' is one of $\mathrm{Aut}(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, respectively, and let F_2 be a biholomorphic map from G_2p onto M'', where M'' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, that transforms G_2 into G'', where G'' is one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively. We will now construct a biholomorphic map $\mathcal{F}$ from M onto $M' \times M''$. For $q \in M$ consider G_2q and let r be the unique point of intersection of G_1p and G_2q. Let $g \in G_1$ be an element such that $r = gp$. Then we set $\mathcal{F}(q) := (F_1(r), F_2(g^{-1}q))$. Clearly, $\mathcal{F}$ is a well-defined diffeomorphism from M onto $M' \times M''$. Since the foliation of M by G_j-orbits is holomorphic for each j, the map $\mathcal{F}$ is in fact holomorphic. By construction, $\mathcal{F}$ transforms G into $G' \times G''$. Thus, we have obtained (i) of the theorem.

Suppose now that $n = 4$ and LG_p^0 is conjugate in U_4 to Sp_2. In this case LG_p acts transitively on directions in $T_p(M)$. Now the result of [GK] gives, as before, that if M is non-compact, it is holomorphically equivalent to one of $\mathbb{B}^4$, $\mathbb{C}^4$, and an equivalence map can be chosen so that it maps p into the origin, transforms G_p into a subgroup of $U_4 \subset G(\mathbb{C}^4)$, and transforms G_p^0 into Sp_2. Furthermore, the result of [BDK] gives, as before, that if M is compact, it is holomorphically equivalent to $\mathbb{CP}^4$.

If M is equivalent to $\mathbb{CP}^4$, arguing as in the proof of Theorem 1.1, we obtain that SU_5 has a closed 18-dimensional subgroup. This contradicts Lemma 2.1 in [IKra] (see also Lemma 1.4 in [I4]), and therefore M cannot be equivalent to $\mathbb{CP}^4$.

Assume now that $n = 4$, the manifold M is equivalent to one of $\mathbb{B}^4$, $\mathbb{C}^4$ and let F be an equivalence map that transforms G_p^0 into $Sp_2 \subset G(\mathbb{C}^4)$. Let $\mathfrak{g}$ be the Lie algebra of fundamental vector fields of the action of $G^F := F \circ G \circ F^{-1}$

on one of $\mathbb{B}^4$, $\mathbb{C}^4$, respectively. Since G^F acts transitively on one of $\mathbb{B}^4$, $\mathbb{C}^4$, the algebra $\mathfrak{g}$ is generated by $\mathfrak{sp}_2$ (where, as before, $\mathfrak{sp}_2$ denotes the Lie algebra of Sp_2 realized as the algebra of fundamental vector fields of the standard action of Sp_2 on $\mathbb{C}^4$), and some vector fields (1.6), where f_j^k, g_j^k, $j, k = 1, 2, 3, 4$ are functions holomorphic on one of $\mathbb{B}^4$, $\mathbb{C}^4$, respectively, and satisfying (1.7). We will show that F maps M onto $\mathbb{C}^4$ and transforms G into $G_3(\mathbb{C}^4)$. To obtain this, it is sufficient to prove that one can choose

$$\begin{array}{rcl} V_j & = & \partial/\partial z_j, \\ W_j & = & i\,\partial/\partial z_j, \end{array}$$

for $j = 1, 2, 3, 4$.

In our arguments we will use the following vector fields from $\mathfrak{sp}_2$: Z_4, Z_5, Z_6, Z_7, Z_8, Z_9 defined in (1.9), as well as the vector fields

$$\begin{array}{rcl} Z_1' & := & iz_2\,\partial/\partial z_2 - iz_4\,\partial/\partial z_4, \\ Z_2' & := & iz_1\,\partial/\partial z_1 - iz_3\,\partial/\partial z_3 \end{array}$$

(observe that Z_1, Z_2, Z_3 defined in (1.9) do not line in $\mathfrak{sp}_2$). It is straightforward to see that $[V_1, Z_1'](0) = 0$, and therefore we have

$$[V_1, Z_1'] = 0 \pmod{\mathfrak{sp}_2}. \tag{2.4}$$

Representing f_1^k by a power series near the origin and denoting by $\tilde{f}_1^k$ the non-linear part of its expansion for $k = 1, 2, 3, 4$, from (2.4) we obtain

$$\begin{array}{rcl} \tilde{f}_1^1 & = & \displaystyle\sum_{n+2l+m\geq 2} a^1_{1\,n,l,m,l} z_1^n z_2^l z_3^m z_4^l, \\ \tilde{f}_1^2 & = & \displaystyle\sum_{n+2l+m\geq 1} a^2_{1\,n,l+1,m,l} z_1^n z_2^{l+1} z_3^m z_4^l, \\ \tilde{f}_1^3 & = & \displaystyle\sum_{n+2l+m\geq 2} a^3_{1\,n,l,m,l} z_1^n z_2^l z_3^m z_4^l, \\ \tilde{f}_1^4 & = & \displaystyle\sum_{n+2l+m\geq 1} a^4_{1\,n,l,m,l+1} z_1^n z_2^l z_3^m z_4^{l+1}, \end{array}$$

where $a^k_{1\,n,l,m,r} \in \mathbb{C}$. Next, we observe

$$\begin{array}{rcl} [V_1, Z_2'](0) & = & (i, 0, 0, 0), \\ [W_1, Z_2'](0) & = & (-1, 0, 0, 0). \end{array}$$

It then follows that

$$\begin{array}{rcl} V_1 & = & -[W_1, Z_2'] \quad (\mathrm{mod}\ \mathfrak{sp}_2), \\ W_1 & = & [V_1, Z_2'] \quad (\mathrm{mod}\ \mathfrak{sp}_2), \end{array} \tag{2.5}$$

which yields

$$V_1 = -[[V_1, Z_2'], Z_2'] \quad (\mathrm{mod}\ \mathfrak{sp}_2). \tag{2.6}$$

Formula (2.6) implies that $\tilde{f}_1^k$, $k = 1, 2, 3, 4$, in fact have the forms

$$\begin{array}{rcl} \tilde{f}_1^1 & = & \displaystyle\sum_{n+l\geq 1} a^1_{1\,n,l,n,l} z_1^n z_2^l z_3^n z_4^l + \sum_{n,l\geq 0} a^1_{1\,n+2,l,n,l} z_1^{n+2} z_2^l z_3^n z_4^l, \\ \tilde{f}_1^2 & = & \displaystyle\sum_{n,l\geq 0} a^2_{1\,n,l+1,n+1,l} z_1^n z_2^{l+1} z_3^{n+1} z_4^l + \sum_{n,l\geq 0} a^2_{1\,n+1,l+1,n,l} z_1^{n+1} z_2^{l+1} z_3^n z_4^l, \\ \tilde{f}_1^3 & = & \displaystyle\sum_{n+l\geq 1} a^3_{1\,n,l,n,l} z_1^n z_2^l z_3^n z_4^l + \sum_{n,l\geq 0} a^3_{1\,n,l,n+2,l} z_1^n z_2^l z_3^{n+2} z_4^l, \\ \tilde{f}_1^4 & = & \displaystyle\sum_{n,l\geq 0} a^4_{1\,n,l,n+1,l+1} z_1^n z_2^l z_3^{n+1} z_4^{l+1} + \sum_{n,l\geq 0} a^4_{1\,n+1,l,n,l+1} z_1^{n+1} z_2^l z_3^n z_4^{l+1}. \end{array}$$

In addition, (2.4) and (2.6) imply that the linear part of V_1 is an element of $\mathfrak{sp}_2$. Subtracting this element from V_1, we can assume that V_1 has no linear part.

Next, we consider $[V_1, Z_4]$. It is easy to see that $[V_1, Z_4](0) = 0$, which yields

$$[V_1, Z_4] = 0 \quad (\mathrm{mod}\ \mathfrak{sp}_2). \tag{2.7}$$

It follows from (2.7) that the forms of $\tilde{f}_1^k$, $k = 1, 2, 3, 4$, further simplify as

$$\begin{array}{rcl} \tilde{f}_1^1 & = & \displaystyle\sum_{n\geq 1} a^1_{1\,n,0,n,0} z_1^n z_3^n + \sum_{n\geq 0} a^1_{1\,n+2,0,n,0} z_1^{n+2} z_3^n, \\ \tilde{f}_1^2 & = & \displaystyle\sum_{n\geq 0} a^2_{1\,n,1,n+1,0} z_1^n z_2 z_3^{n+1} + \sum_{n\geq 0} a^2_{1\,n+1,1,n,0} z_1^{n+1} z_2 z_3^n, \\ \tilde{f}_1^3 & = & \displaystyle\sum_{n\geq 1} a^3_{1\,n,0,n,0} z_1^n z_3^n + \sum_{n\geq 0} a^3_{1\,n,0,n+2,0} z_1^n z_3^{n+2}, \\ \tilde{f}_1^4 & = & \displaystyle\sum_{n\geq 0} a^4_{1\,n,0,n+1,1} z_1^n z_3^{n+1} z_4 + \sum_{n\geq 0} a^4_{1\,n+1,0,n,1} z_1^{n+1} z_3^n z_4. \end{array}$$

Applying the above arguments to V_3 in place of V_1 we obtain that the linear part of V_3 at the origin is an element of $\mathfrak{sp}_2$ and that the non-linear parts $\tilde{f}_3^k$ of the expansions around the origin of the functions f_3^k, $k = 1, 2, 3, 4$, have the following forms

$$\begin{aligned} \tilde{f}_3^1 &= \sum_{n\geq 1} a^1_{3\,n,0,n,0} z_1^n z_3^n + \sum_{n\geq 0} a^1_{3\,n+2,0,n,0} z_1^{n+2} z_3^n, \\ \tilde{f}_3^2 &= \sum_{n\geq 0} a^2_{3\,n+1,1,n,0} z_1^{n+1} z_2 z_3^n + \sum_{n\geq 0} a^2_{3\,n,1,n+1,0} z_1^n z_2 z_3^{n+1}, \\ \tilde{f}_3^3 &= \sum_{n\geq 1} a^3_{3\,n,0,n,0} z_1^n z_3^n + \sum_{n\geq 0} a^3_{3\,n,0,n+2,0} z_1^n z_3^{n+2}, \\ \tilde{f}_3^4 &= \sum_{n\geq 0} a^4_{3\,n+1,0,n,1} z_1^{n+1} z_3^n z_4 + \sum_{n\geq 0} a^4_{3\,n,0,n+1,1} z_1^n z_3^{n+1} z_4, \end{aligned} \tag{2.8}$$

where $a^k_{3\,n,l,m,r} \in \mathbb{C}$. Next, we observe

$$[V_1, Z_8](0) = (0, 0, -1, 0),$$

which gives

$$V_3 = -[V_1, Z_8] \pmod{\mathfrak{sp}_2}. \tag{2.9}$$

Formulas (2.8) and (2.9) imply

$$\begin{aligned} V_1 \;=\; &(1 + \alpha z_1^2 + a z_1 z_3)\,\partial/\partial z_1 + (\beta z_1 z_2 + b z_2 z_3)\,\partial/\partial z_2 + \\ &(\alpha z_1 z_3 + a z_3^2)\,\partial/\partial z_3 + (\varepsilon z_1 z_4 + c z_3 z_4)\,\partial/\partial z_4, \end{aligned} \tag{2.10}$$

for some $a, b, c, \alpha, \beta, \varepsilon \in \mathbb{C}$ (cf. (1.10)). Plugging this expression into (2.7) yields

$$\varepsilon = \beta, \quad c = b. \tag{2.11}$$

Further, if in the above argument we replace identity (2.6) by the identity

$$W_1 = -[[W_1, Z_2'], Z_2'] \pmod{\mathfrak{sp}_2}$$

(which is also a consequence of (2.5)) and consider W_3 instead of V_3, we obtain that W_1 can be chosen to have the form

$$\begin{aligned} W_1 \;=\; &(i + \alpha' z_1^2 + a' z_1 z_3)\,\partial/\partial z_1 + (\beta' z_1 z_2 + b' z_2 z_3)\,\partial/\partial z_2 + \\ &(\alpha' z_1 z_3 + a' z_3^2)\,\partial/\partial z_3 + (\varepsilon' z_1 z_4 + c' z_3 z_4)\,\partial/\partial z_4. \end{aligned} \tag{2.12}$$

for some $a', b', c', \alpha', \beta', \varepsilon' \in \mathbb{C}$ (cf. (1.11)). Plugging (2.10), (2.12) into either of identities (2.5) we obtain

$$\begin{array}{l} \alpha' = -i\alpha,\ \beta' = -i\beta,\ \varepsilon' = -i\varepsilon, \\ a' = ia,\ b' = ib,\ c' = ic. \end{array} \tag{2.13}$$

Then

$$\begin{aligned}[V_1, W_1] \;\; = \;\; -2i\Big(2\alpha z_1\, \partial/\partial z_1 + \Big(\beta z_2 + (a\beta - b\alpha) z_1 z_2 z_3\Big)\, \partial/\partial z_2 + \\ \alpha z_3\, \partial/\partial z_3 + \Big(\varepsilon z_4 + (a\varepsilon - c\alpha) z_1 z_3 z_4\Big)\, \partial/\partial z_4\Big).\end{aligned}$$

Therefore, $[V_1, W_1] = 0 \,(\mathrm{mod}\ \mathfrak{sp}_2)$, which can only hold if

$$\alpha = 0,\ \varepsilon = -\beta.$$

Together with (2.11) this implies

$$\beta = \varepsilon = 0,$$

hence we have

$$V_1 = (1 + a z_1 z_3)\, \partial/\partial z_1 + b z_2 z_3\, \partial/\partial z_2 + a z_3^2\, \partial/\partial z_3 + b z_3 z_4\, \partial/\partial z_4. \tag{2.14}$$

If in the above arguments we interchange Z_1', Z_2', as well as Z_4, Z_8, and use V_2 in place of V_1, W_2 in place of W_1, V_4 in place of V_3, and W_4 in place of W_3, we obtain that V_2 can be chosen to have the form

$$V_2 = d z_1 z_4\, \partial/\partial z_1 + (1 + e z_2 z_4)\, \partial/\partial z_2 + d z_3 z_4\, \partial/\partial z_3 + e z_4^2\, \partial/\partial z_4, \tag{2.15}$$

for some $d, e \in \mathbb{C}$. We will now consider $[V_1, Z_6]$. It is straightforward to see that

$$[V_1, Z_6](0) = (0, -1, 0, 0),$$

and therefore we have

$$V_2 = -[V_1, Z_6] \quad (\mathrm{mod}\ \mathfrak{sp}_2). \tag{2.16}$$

Formulas (2.15), (2.16) imply

$$a = b = d = e. \tag{2.17}$$

We then compute

$$[V_1, V_2] = a\,(z_4\,\partial/\partial z_1 - z_3\,\partial/\partial z_2)\,.$$

Therefore $[V_1, V_2] = 0\,(\mathrm{mod}\ \mathfrak{sp}_2)$, which can only hold if $a = 0$. Hence it follows from (2.12), (2.13), (2.14), (2.15), (2.17) that

$$\begin{array}{rcl} V_1 & = & \partial/\partial z_1, \\ W_1 & = & i\,\partial/\partial z_1, \\ V_2 & = & \partial/\partial z_2. \end{array}$$

Therefore identities (1.17) hold, and we obtain

$$\begin{array}{rcll} W_2 & = & i\,\partial/\partial z_2 & (\mathrm{mod}\ \mathfrak{sp}_2), \\ V_3 & = & \partial/\partial z_3 & (\mathrm{mod}\ \mathfrak{sp}_2), \\ W_3 & = & i\,\partial/\partial z_3 & (\mathrm{mod}\ \mathfrak{sp}_2), \\ V_4 & = & \partial/\partial z_4 & (\mathrm{mod}\ \mathfrak{sp}_2), \\ W_4 & = & i\,\partial/\partial z_4 & (\mathrm{mod}\ \mathfrak{sp}_2). \end{array}$$

Hence W_2, V_3, W_3, V_4, W_4 can be chosen as in formula (1.18), and we have obtained (ii) of the theorem.

The proof is now complete. □

Remark 2.2 In the situations arising in Part (ii) of Theorem 1.1 and in both parts of Theorem 2.1 the group LG_q^0 contains the map $-\mathrm{id}$ for every $q \in M$. Therefore, M equipped with a G-invariant Hermitian metric becomes a Hermitian symmetric space. Then, with some extra work, Part (ii) of Theorem 1.1 as well as all of Theorem 2.1 follow from E. Cartan's classification of Hermitian symmetric spaces (see [H]). The same applies to Part (i) of Theorem 1.1 if n is even. We also remark that Part (i) of Theorem 1.1 for all n follows from the results of [Wo] (see Theorem 13.1 therein). Our proofs of Theorems 1.1 and 2.1 given above are elementary and do not refer to this general theory.

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